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A formula for Nijenhuis torsion and the uniqueness aspect of the inverse problem of Lagrangian dynamics

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Abstract. A formula is derived for the Nijenhuis tensor of an endomorphism constructed from contracting a bivector field with a 2-form. Two applications are considered: the first is to the uniqueness aspect of the inverse problem of the calculus of variations, the second is to bi-Hamiltonian systems.

1. Introduction

Let Ω and ω be type $(2,0)$ and $(0,2)$ skew-symmetric tensor fields on a smooth manifold M. Further, let R be the one-stage contraction of Ω with ω, that is, if X is a vector field on M

$$RX = \Omega(-, i(X)\omega). \tag{1.1}$$

The following problem occurs in several contexts in theoretical mechanics: to compute the Nijenhuis tension N_R in terms of Ω and ω. We shall solve this problem in proposition 2.2.

Two of the situations alluded to above are as follows.

(1) Let M be the tangent bundle of a differentiable manifold N and let Γ be a second-order vector field on TN. Suppose that Γ is an Euler–Lagrange vector field and that Γ has two Cartan 2-forms that we denote by Ω^{-1} and ω. Define an endomorphism field R according to (1.1) above.

(2) Let M be a manifold endowed with two Poisson structures Ω and ω^{-1}. As the notation suggests, the second Poisson structure is non-degenerate and so ω is a symplectic form. A vector field X with the property that there exist functions H_0 and H_1 on M such that

$$X = \Omega(-, \mathrm{d}H_0) = \omega^{-1}(-, \mathrm{d}H_1) \tag{1.2}$$

is called a bi-Hamiltonian system. (For more details see [10].) The Poisson structures are said to be *compatible* if for all choices of constants λ_0 and λ_1, $\lambda_0\Omega + \lambda_1\omega^{-1}$ is also a Poisson structure. We shall reconsider this compatibility condition in section 4.

The inverse problem of Lagrangian dynamics is concerned with finding necessary and sufficient conditions for a second-order vector field Γ on a tangent bundle TN to be the Euler–Lagrange vector field of a regular Lagrangian L. We shall limit our attention to the autonomous case but clearly the results may be extended to the time-dependent case.

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For details about tangent bundle geometry and the invariant formulation of Lagrangian dynamics, we refer the reader to Crampin [3] and Morandi *et al* [8] and adopt the notation of [3].

The following answer to the inverse problem was given by Crampin [2] and provides a geometrization of the well known Helmholtz conditions.

Theorem. Necessary and sufficient conditions for Γ to be derivable from a regular Lagrangian are that there exists on TM a 2-form ω, of maximal rank, for which $L_\Gamma\omega = 0$, and such that all vertical subspaces are Lagrangian both for ω and $(i)H\mathrm{d}\omega$ where H is any horizontal vector.

In particular, the Helmholtz conditions do not demand *a priori* that ω should be closed.

The other aspect of the inverse problem is whether the Lagrangian, if it exists, is essentially unique, Several authors have argued that the existence of distinct Lagrangians, and hence an endomorphism field R, is indicative of the complete integrability of the dynamical system Γ [3, 4, 8]. Obviously the existence of alternative Lagrangians closely resembles the theory of a bi-Hamiltonian system. However, one should not imagine that the two theories are simply related. In fact under Legendre transform Γ will, in general, be mapped to two quite distinct Hamiltonian vector fields on T^*N depending on which Lagrangian is used.

Although bi-Hamiltonian systems have probably attracted more attention than bi-Lagrangian systems [6], there are three works [3, 4, 8] on the latter which we shall cite and which it is our intention to elucidate and supplement. The point of view adopted in [3, 4, 8] is to suppose that a second-order field Γ possesses two essentially distinct Lagrangians, at least one of which is regular. Using the inverse of the Cartan 2-form of the regular Lagrangian and the Cartan 2-form of the other Lagrangian, one may construct the recursion operator R in the manner of equation (1.1). One then shows that constants of motion may be constructed from traces of powers of R under the assumption that the Nijenhuis torsion N_R is zero. Our main theorems below clarify the relationship between the existence of alternative Cartan forms and the vanishing of N_R. Conversely, we formulate necessary and sufficient conditions for an endomorphism field R, having certain properties including the vanishing of N_R, to engender an alternative Cartan form.

The final point in this introductory section concerns the somewhat contentious question of notation. One of the drawbacks of the coordinate-free formulation of modern differential geometry is that there is no convenient way to denote a contraction of contravariant and convariant tensors whose rank is larger that one. Of course, in the presence of a bivector field Ω and 2-form ω we have musical morphisms $\#_\Omega : T^*M \to TM$ and $\flat_\omega : TM \to T^*M$, respectively. Here, if α is a 1-form on M, $\#_\omega(\alpha)$ or $\alpha^\#$ is defined by $\beta(\alpha^\#) = \Omega(\alpha, \beta)$ for all 1-forms β on M. Similarly for a vector field X on M, $\flat_\omega(X)$ or $X^\flat$ is defined as $i(X)\omega$. However, these constructions are again most useful for changing 1-forms into vector fields and *vice versa* although one can induce mappings on higher rank tensor bundles. The musical morphisms are used occasionally below, # and $\flat$ always defined relative to Ω and ω, respectively. In order to avoid an excessively cluttered notation we have often resorted to a dot to denote a contraction. The reader should bear in mind that throughout the paper all contractions are 'one stage'; that is, adjacent tensors separated by a dot are contracted over adjacent 'indices' only. For example, if α is a 1-form, the vector field $\Omega \cdot \alpha$ and 2-form $\omega \cdot \Omega \cdot \omega$ would be written locally as $\Omega^{ij}\alpha_j$ and $\omega_{ij}\Omega^{jk}\omega_{kl}$, respectively.

2. A formula for the Nijenhuis torsion of R

In this section we shall derive a formula for N_R where R is defined by (1.1). We begin with a technical lemma in which a 1-form on M is denoted by α, the vector field dual to α via Ω by $\alpha^\#$ and the Lie derivative operator by L. Also, $[\Omega, \Omega]$ denotes the Schouten concomitant of Ω with itself, for details of which we refer to Nijenhuis' original paper [9]. See also [6, 7].

Lemma 2.1.

$$L_{\alpha^\#}\Omega = [\Omega \cdot \alpha, \Omega] = 1/4[\Omega, \Omega] \cdot \alpha + \Omega \cdot \mathrm{d}\alpha \cdot \Omega. \tag{2.1}$$

Proof. It is enough to establish the result for Ω of the form $X \wedge Y$ because of the linearity properties of the Schouten bracket. Thus

$$\begin{aligned}2[\Omega \cdot \alpha, \Omega] &= L_{\langle Y,\alpha\rangle X - \langle X,\alpha\rangle Y} X \wedge Y\\ &= -\langle Y, \alpha\rangle[X, Y] \wedge X + \langle X, \alpha\rangle[X, Y] \wedge Y - (X\langle Y, \alpha\rangle - Y\langle X, \alpha\rangle)X \wedge Y\\ &= -\langle Y, \alpha\rangle[X, Y] \wedge X + \langle X, \alpha\rangle[X, Y] \wedge Y - \langle[X, Y], \alpha\rangle X \wedge Y\\ &\quad - \mathrm{d}\alpha(X, Y)X \wedge Y.\end{aligned}$$

However, it is easy to check with Ω of the form $X \wedge Y$ that $\Omega \cdot \mathrm{d}\alpha \cdot \Omega$ is just $-\frac{1}{2}\mathrm{d}\alpha(X, Y)X \wedge Y$. Finally following [6] we see that

$$[\Omega, \Omega] = -2X \wedge Y \wedge [X, Y]$$

and hence we obtain the result. □

Proposition 2.2.

$$\begin{aligned}N_R(X, Y) = \tfrac{1}{4}[\Omega, \Omega](X^\flat, Y^\flat, -) + [&i(Y)i(X)\,\mathrm{d}(\omega \cdot \Omega \cdot \omega) - \mathrm{d}\omega(RX, Y, -)\\ &+\mathrm{d}\omega(RY, X, -)]^\#\end{aligned} \tag{2.2}$$

Proof. Recall that the definition of N_R is given by

$$N_R(X, Y) = [RX, RY] - R[RX, Y] - R[X, RY] + R^2[X, Y]. \tag{2.3}$$

We rewrite N_R in the form

$$N_R(X, Y) = R \cdot (L_Y R)(X) - (L_{RY} R)(X) \tag{2.4}$$

noting that the tensorial property of N_R is thus made evident. From now on we shall suppress the X variable and recalling that R is $\Omega \cdot \omega$ we obtain

$$N_R(-Y) = \Omega \cdot \omega \cdot L_Y\Omega \cdot \omega + \Omega \cdot \omega \cdot \Omega \cdot L_Y\omega - (L_{RY}\Omega) \cdot \omega - \Omega \cdot L_{RY}\omega. \tag{2.5}$$

We rewrite the third term in (2.5) by means of lemma 2.1, with α being the 1-form $-i(Y)\omega$, as $[1/4[\Omega, \Omega] \cdot i(Y)\omega \cdot +\Omega \cdot d(i(Y)\omega) \cdot \Omega] \cdot \omega$. For the moment we shall ignore the Schouten bracket term and notice that all other terms lead off with an Ω, which we shall ignore. What remains is, on replacing $\mathrm{d}(i(Y)\omega)$ by $L_Y\omega - i(Y)\,\mathrm{d}\omega$:

$$\begin{aligned}&\omega \cdot L_Y\Omega \cdot \omega + \omega \cdot \Omega \cdot L_Y\omega + L_Y\omega \cdot \Omega \cdot \omega - i(Y)\,\mathrm{d}\omega \cdot \Omega \cdot \omega\\ &\qquad -i(RY)\,\mathrm{d}\omega - \mathrm{d}(i(RY)\omega).\end{aligned} \tag{2.6}$$

The first three terms may be rewritten as $L_Y(\omega \cdot \Omega \cdot \omega)$ and hence (2.6) may be recast as

$$\begin{aligned}&i(Y)\,\mathrm{d}(\omega \cdot \Omega \cdot \omega) + \mathrm{d}(i(Y)\omega \cdot \Omega \cdot \omega) - i(Y)\,\mathrm{d}\omega \cdot \Omega \cdot \omega - i(\Omega \cdot \omega(Y))\,\mathrm{d}\omega\\ &\qquad - \mathrm{d}(i(\Omega \cdot \omega(Y))\omega).\end{aligned} \tag{2.7}$$

However, we see that the second and fifth terms in (2.7) are equal but for sign and so cancel. Finally going back to (2.5) and putting everything together we obtain the formula (2.2) stated above. □

We should mention also that formula (2.2) continues Nijenhuis' program of relating the various differential concomitants introduced by himself and Schouten [9].

3. The uniqueness aspect of the Lagrangian inverse problem

We turn now to applications of formula (2.2) and begin with the uniqueness aspect of the Lagrangian inverse problem. Theorem 3.2 below elucidates the Nijenhuis condition for a recursion operator constructed from a pair of Cartan 2-forms. The proof depends on the following lemma which is based on [2] and [8]. The canonical almost tangent structure on TN is denoted by S.

Lemma 3.1. Let Γ be a second-order vector field that is of Euler–Lagrange type, with Ω^{-1} being its Cartan 2-form. suppose that ω is an alternative not necessarily non-degenerate Cartan 2-form for Γ and that the recursion operator R is defined by equation (1.1). Then R satisfies:

(i) $\omega(RX, Y) = \omega(X, RY)$, for all vector fields X, Y on TN;
(ii) $R \cdot S = S \cdot R$;
(iii) $R \cdot L_\Gamma S = L_\Gamma S \cdot R$;
(iv) the horizontal and vertical distributions of Γ and R-invariant.

Proof. (i) For arbitrary X and Y we have

$$\begin{aligned}\omega(RX, Y) &= \omega(\Omega \cdot \omega(X), Y) = \Omega(i(Y)\omega, i(X)\omega) = -\Omega(i(X)\omega, i(Y)\omega)\\ &= -\omega(RY, X) = \omega(X, RY).\end{aligned}$$

(ii) Let Z and W be vector fields on N and Z^{V}, W^{V} and Z^{h}, W^{h} their vertical and horizontal lifts, respectively. Since vertical subspaces are Lagrangian for ω and $L_\Gamma\omega$ is zero we find that

$$\omega([\Gamma Z^{\mathrm{V}}], W^{\mathrm{V}}) + \omega(Z^{\mathrm{V}}, [\Gamma, W^{\mathrm{V}}]) = 0. \tag{3.1}$$

Again, because vertical subspaces are Lagrangian, we may replace $[\Gamma, Z^{\mathrm{V}}]$ by its horizontal component $-Z^{\mathrm{h}}$ and likewise for W. Thus

$$\omega(Z^{\mathrm{h}}, W^{\mathrm{V}}) = \omega(W^{\mathrm{h}}, Z^{\mathrm{V}}). \tag{3.2}$$

The last equation easily implies that

$$\omega(SX, Y) = \omega(SY, X) \tag{3.3}$$

for all X and Y. Now

$$\Omega^{-1}(RSX, Y) = \omega(SX, Y) = -\omega(X, SY) = -\Omega^{-1}(RX, SY) = \Omega^{-1}(SRX, Y).$$

Hence since Ω^{-1} is non-degenerate and X and Y are arbitrary $RS = SR$.

(iii) is immediate from (ii) and the fact that $L_\Gamma R = 0$.

(iv) follows from (iii) and the fact that the horizontal projector of Γ is $1/2\,[I - L_\Gamma S]$. □

Theorem 3.2. Let Γ be a second-order vector field that is of Euler–Lagrange type and let Ω^{-1} be its Cartan 2-form. Suppose that R is a non-singular endomorphism field on TN that satisfies the following conditions:

(i) $\Omega^{-1} \cdot R = \omega$, say, is skew-symmetric (and of type (0, 2));
(ii) The vertical distribution on TN is invariant by R;
(iii) $L_\Gamma R = 0$;

(iv) the Nijenhuis tensor N_R of R vanishes. Then ω is an alternative Cartan 2-form for Γ if and only if

$$\Omega^{-1}(W, (L_V R)H) - \Omega^{-1}(V, (L_W R)H) = 0 \tag{3.4}$$

where H is an arbitrary horizontal and V and W arbitrary vertical vector fields.

Proof. We show that ω satisfies the conditions of Crampin's theorem. Of course Ω^{-1} satisfies those conditions and so, because of (iii), $L_\Gamma\omega = 0$. Next condition (ii) above and the fact that vertical subspaces are Lagrangian for Ω^{-1} implies the same of ω.

Finally we turn to the condition involving horizontal vectors and consider condition (iv). Now ω is of course a Poisson tensor and so the vanishing of N_R may be written as, in view of proposition 2.2.

$$i(H)i(W)i(V)\,\mathrm{d}(\omega\cdot\Omega\cdot\omega) - \mathrm{d}\omega(RV, W, H) + \mathrm{d}\omega(RW, V, H) = 0 \tag{3.5}$$

where V, W and H are arbitrary vector fields on TN.

Now let us specialize, as the notation suggests, to the case where V and W are vertical and H is horizontal. We compute each of the three exterior derivatives using the six-term formula taking into account the following considerations: three of the 18 terms are zero as a result of the fact that vertical subspaces are Lagrangian for ω and that the horizontal distribution if R-invariant by the argument of lemma 3.1 (iv). Four further pairs of terms cancel because of the switching property of lemma 3.1 (i) so that (3.5) reduces to

$$\begin{aligned} &RV(\omega(H, W)) - RW(\omega(H, V)) + \omega(H, R[V, W] - [RV, W] - [V, RW]) \\ &\qquad + \omega(W, [RV, H] - \omega(V, [RW, H]) = 0. \end{aligned} \tag{3.6}$$

On the other hand, the last condition in Crampin's theorem requires of ω that

$$V(\omega(W, H)) + W(\omega(H, V)) + \omega(H, [V, W]) + \omega(V, [W, H]) + \omega(W, [H, V]) = 0 \tag{3.7}$$

where $H(\omega(V, W))$ vanishes in (3.7) because of the Lagrangian condition satisfied by ω.

Since R is assumed to be non-singular and leaves the horizontal and vertical distributions invariant we may replace H by RH and also use the vanishing of N_R as given by (2.3) to simplify (3.6) further, thereby obtaining

$$\begin{aligned} &RV(\omega(RW, H)) + RW(\omega(H, RV)) + \omega(W, [RH, RV]) + \omega(V, [RW, RH]) \\ &\qquad + \omega(H, [RV, RW]) = 0. \end{aligned} \tag{3.8}$$

Similarly in (3.7) we replace V and W by RV and RW, respectively:

$$\begin{aligned} &RV(\omega(RW, H)) + RW(\omega(H, RV)) + \omega(W, R[H, RV]) + \omega(V, R[RW, H]) \\ &\qquad + \omega(H, [RV, RW]) = 0. \end{aligned} \tag{3.9}$$

Clearly (3.8) and (3.9) are equivalent if and only if

$$\omega(W, (L_{RV} R)H) - \omega(V, (L_{RW} R)H) = 0. \tag{3.10}$$

Finally, eliminating ω in favour of Ω and using the fact that RV and RW are arbitrary, vertical vector fields (3.10) may be recast in the form (3.4). □

The following theorem provides something of a converse to theorem 3.2

Theorem 3.3. Let Γ be a second-order vector field that is of Euler–Lagrange type and let Ω^{-1} be its Cartan 2-form. Suppose further that ω is a not necessarily non-degenerate alternative Cartan 2-form for Γ. Then if R and N_R are defined by (1.1) and (2.3), respectively, then N_R is zero if and only if

$$\omega(V, (L_W R)H) - \omega(W, (L_V R)H) = 0 \tag{3.11}$$

where H is a horizontal and V and W are vertical vector fields.

Proof. Clearly according to proposition 2.2 we have that N_R is zero if and only if the 2-form $\omega \cdot R$ is closed. However, we have that $L_\Gamma(\omega \cdot R) = 0$ and the vertical distribution is Lagrangian for $\omega \cdot R$ by lemma 3.1 (iv). Thus according to Crampin's theorem $\omega \cdot R$ is closed if and only if

$$V(\omega(W, RH)) + W(\omega(H, RV)) + \omega(H, R[V, W]) + \omega(V, R[W, H]) \\ +\omega(W, R[H, V]) = 0 \tag{3.12}$$

where H is a horizontal and V and W are vertical vector fields, respectively. However, if we evaluate $\mathrm{d}\omega(RH, V, W)$ we obtain, making use of lemma 3.1(i),

$$V(\omega(W, RH)) + W(\omega(H, RV)) + \omega(H, R[V, W]) + \omega(V, [W, RH]) \\ +\omega(W, [RH, V]) = 0. \tag{3.13}$$

Clearly, however, (3.11) is the necessary and sufficient condition for the equality of (3.12) and (3.13). □

We conclude this section with two examples that are intended to illustrate the scope of theorems 3.2 and 3.3.

Example 1. Let g be a metric of any signature on a manifold N of dimension n. Locally g is represented by the matrix g_{ij} relative to a coordinate system (x^i) on N. The complete lift of g denoted by g^c is a metric on TN of signature (n, n) and in induced coordinates (x^i, u^i) on TN corresponds to the metric $\begin{bmatrix} 0 & g \\ g & \dot{g} \end{bmatrix}$ where the dot denotes total time derivative. For more details on the complete lift construction we refer to [11]. The pullback of g to TN is parallel with respect to the Levi-Civita connection ∇^{c} of g^{c}. Passing now to TTN, g^{c} and g induce non-degenerate and degenerate Lagrangians, respectively. The resulting recursion operator R is parallel with respect to ∇^{c} and hence N_R is zero.

Example 2. We consider the second-order vector field Γ defined by

$$\Gamma = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + y\frac{\partial}{\partial u} \tag{3.14}$$

where u and v stand for $\dot{x}$ and $\dot{y}$, respectively. The corresponding system of second-order ordinary differential equation is one of the examples considered in the famous article of Douglas [5] on the inverse problem. It is also considered in some detail in [1].

Notice that we have chosen to include t in (3.14). The reason is that following [1] both the Lagrangians L_1 and L_2 engender (3.14), where

$$L_1 = uv + \tfrac{1}{2}y^2 \tag{3.15}$$

$$L_2 = uv^2 - tyv^2 + \tfrac{1}{6}t^2v^3. \tag{3.16}$$

We leave it to the reader to check, allowing for the occurrence of t and the obvious modifications necessitated thereby, that the Nijenhuis tensor of the recursion operator defined by L_1 and L_2 is not zero. We now have three ways to do this: (i) construct N_R directly, (ii) use Theorem 3.3 or rather its time-dependent analogue, (iii) check if the form $\omega \cdot R$ is closed.

4. A criterion for bi-Hamiltonian structures

Finally let us reconsider the bi-Hamiltonian systems introduced above and derive the following result.

Proposition 4.1. The Poisson structures Ω and ω^{-1} are compatible if and only if the 2-form $\omega \cdot \Omega \cdot \omega$ is closed.

Proof. First of all note that the Schouten bracket of $\lambda_0\Omega + \lambda_1\omega^{-1}$ with itself is given by

$$[\lambda_0\Omega + \lambda_1\omega^{-1}, \lambda_0\Omega + \lambda_1\omega^{-1}] = \lambda_0^2[\Omega, \Omega] + \lambda_0\lambda_1[\Omega, \omega^{-1}] + \lambda_0\lambda_1[\omega^{-1}, \Omega] + \lambda_1^2[\omega^{-1}, \omega^{-1}]$$
$$= 2\lambda_0\lambda_1[\Omega, \omega^{-1}].$$

Now consider the following equality of endomorphism fields where R is defined by equation (1.1):

$$\lambda_0 R + \lambda_1 I = (\lambda_0\Omega + \lambda_1\omega^{-1}) \cdot \omega. \tag{4.1}$$

We evaluate the Nijenhuis tensor of the left-hand side directly from (2.3) and that of the right-hand side from (2.2), using the formula for the Schouten bracket of $\lambda_0\Omega + \lambda_1\omega^{-1}$ with itself previously obtained. In fact, we find that after dividing by $\lambda_0\lambda_1$, X and Y being arbitrary vector fields,

$$[\Omega, \omega^{-1}] \cdot i(X)\omega \wedge i(Y)\omega = \omega^{-1} \cdot i(Y)i(X) \cdot \mathrm{d}(\omega \cdot \Omega \cdot \omega). \tag{4.2}$$

From (4.2) we see that vanishing of the Schouten bracket $[\Omega, \omega^{-1}]$ is equivalent to the closure of the 2-form $\omega \cdot \Omega \cdot \omega$. $\square$

In conclusion we might mention that proposition 4.1 is quite practical because it is usually easier to check the closure of a form than to calculate a Schouten bracket.

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